5 Calculus

5.1 Introduction to Differentiation

Calculus provides a method of solving many real world problems that cannot be solved with algebra alone. It is the basis of essential mathematics in areas such as engineering, medicine and economics. Calculus allows us to work with situations where a quantity is continuously changing, a very useful tool given that most mathematical problems that we wish to solve will involve change of some description.

Differentiation is a technique used to calculate the gradient, or slope, of a graph at different points. It also allows us to calculate the rate of change of one variable with respect to another. Given a function, for example, $y = x^2$, it is possible to derive a formula for the gradient of the graph. This will give us the gradient function because it will tell us the gradient at any point on the curve. In our example, $y = x^2$, the gradient function is $2x$. From this we can see that the gradient of the graph of $y = x^2$ at any point on the curve is twice the x value there. For example, when $x = 5$, the gradient is $2 \times 5 = 10$. When $x = -4$, the gradient is $2 \times -4 = -8$.

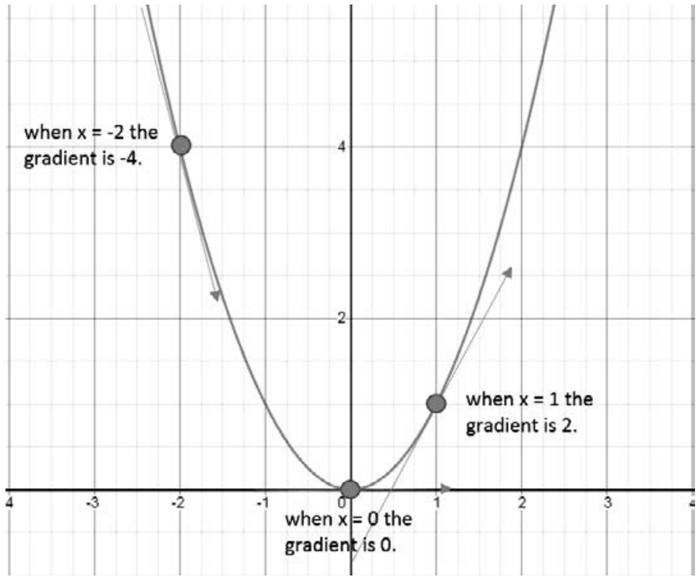
5.1.1 What do these numbers mean?

A gradient of 10 tells us that values of y are increasing at the rate of 10 units for every 1 unit increase in x.

A gradient of −8 tells us that values of y are decreasing at the rate of 8 units for every 1 unit increase in x.

Below is the graph of the function $y = x^2$. You will notice when $x = 1$ the graph has a postive gradient. When $x = -2$ the graph has a negative gradient. Note how these properties of the gradient can be predicted from the knowledge that the gradient function is $2x$.

What we are actually doing when differentiating is drawing a tangent to the curve at the point we are interested in. Recall that a tangent just touches a curve and does not cross it. When we calculate the gradient we are actually calculating the gradient of the tangent to the function at that point, which is the same as the gradient of the function at that point.



5.1.2 Notation

The most common notation for the gradient function is as follows:

- If y is a function of x, that is $y = f(x)$, the gradient function can be written as dy/dx.
- dy/dx pronounced 'dee y by dee x', is not a fraction even though it might look like one! Think of it as the symbol for the gradient function of $y = f(x)$

The process of finding dy/dx is called **differentiation** with respect to x.

We can differentiate with respect to any letter. For example, if we have a function t, i.e. $y = f(t)$, then the derivative would be written as $\frac{dy}{dt}$

5.2 Differentiating Various Functions

The table below summarises the most common functions and their derivatives:

Function – y	a	ax	ax^n	e^x	e^{ax}	be^{ax}	$\ln(x)$	$\ln(ax)$
Derivative – $\frac{dy}{dx}$	0	a	anx^{n-1}	e^x	ae^{ax}	bae^{ax}	$\frac{1}{x}$	$\frac{1}{x}$

Where a and b are constants. If we have a function that is the sum or difference of two or more functions, for example $y = x^2 \pm e^x$ then the derivative is simply the sum or difference of the derivative of the separate functions, i.e. $\frac{dy}{dx} = 2x \pm e^x$.

Example 5.2.1

Differentiate the function $y = x^3 + 2x^2 + x + 2$.

To differentiate this function we must differentiate each term in turn, i.e.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(2x^2\right) + \frac{d}{dx}\left(x\right) + \frac{d}{dx}\left(2\right)\\ &= \frac{d}{dx}\left(x^3\right) + 2\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(x\right) + \frac{d}{dx}\left(2\right)\\ &= 3x^2 + 2 \times 2x + 1 + 0\\ &= 3x^2 + 4x + 1\end{aligned}$$

Example 5.2.2

Differentiate the function $y = \ln(3x) + e^{2x}$.

To differentiate this function we must differentiate each term in turn, i.e.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\ln(3x)\right) + \frac{d}{dx}\left(e^{2x}\right)\\ &= \frac{1}{x} + 2e^{2x}\end{aligned}$$

Example 5.2.3

Calculate the gradient of the function $y = x^3 - 4x^2 + 3x - 5$ at the point $(3, -5)$.

To differentiate the function we are required to differentiate each term in turn:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^3\right) - \frac{d}{dx}\left(4x^2\right) + \frac{d}{dx}\left(3x\right) - \frac{d}{dx}\left(5\right)\\ &= \frac{d}{dx}\left(x^3\right) - 4\frac{d}{dx}\left(x^2\right) + 3\frac{d}{dx}\left(x\right) - \frac{d}{dx}\left(5\right)\\ &= 3x^2 - 4 \times 2x + 3 \times 1 - 0\\ &= 3x^2 - 8x + 3\end{aligned}$$

Once we have found the gradient function, we can evaluate the gradient at the given point by substituting in the value of x into the derivative to find:

$$\frac{d}{dx} = 3(3)^2 - 8(3) + 3 = 6$$

Confidence Builder Questions Set 5.1

1. Differentiate the following functions with respect to the variable given in brackets:

(a) $y = 2x^3 + 4x^2 + 2x - 1$	(x)	(b) $y = e^{-2x} + 2$	(x)
(c) $y = \ln(5t) + t^3 + 2t^2 - 1$	(t)	(d) $y = -2\ln(t) - 3e^{2t} - 4t$	(t)
(e) $y = x^2 + 3x + 2$	(x)	(f) $y = \ln(2x) + e^x$	(x)
(g) $y = 3e^{4t} + 2t^2$	(t)	(h) $y = \ln(3t) + e^{9t} - 4$	(t)
(i) $y = \frac{1}{x^2} + \sqrt{x} + 2$	(x)	(j) $y = \frac{2}{x^3} + 4x^2 - 2x + 3$	(x)

2. Calculate the value of the gradient of $y = x^3 + 2x^2 + 1$ at the point $(2, 17)$.
3. Calculate the value of the gradient of $y = e^{2x} + 1$ at the point $(0, 2)$.

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